

Topological Coherent Modes for Nonlinear Schrödinger Equation

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Abstract

Nonlinear Schrödinger equation, complemented by a confining potential, possesses a discrete set of stationary solutions. These are called coherent modes, since the nonlinear Schrödinger equation describes coherent states. Such modes are also named topological because the solutions corresponding to different spectral levels have principally different spatial dependences. The theory of resonant excitation of these topological coherent modes is presented. The method of multiscale averaging is employed in deriving the evolution equations for resonant guiding centers. A rigorous qualitative analysis for these nonlinear differential equations is given. Temporal behaviour of fractional populations is illustrated by numerical solutions.

1 Introduction

Nonlinear Schrödinger equation has recently attracted a great deal of interest, since it provides an adequate description for collective quantum states of trapped Bose atoms (see reviews [1–3]). This equation, as applied to Bose systems, is often termed the Gross-Pitaevskii equation, who first considered that physical application [4,5].

For the purpose of accuracy in terminology, it is worth mentioning the following. As is possible to show [6], the nonlinear Schrödinger equation is an *exact equation for coherent states*. While the correct meaning of the Gross-Pitaevskii equation [2] is that of an *approximate mean-field equation for the broken symmetry order parameter*.

Coherent states of trapped atoms are the solutions of the nonlinear Schrödinger equation with a confining potential. Stationary solutions of this equation form a discrete set. Wave functions, corresponding to different energy levels, are called [6] *coherent modes*. Different coherent modes possess qualitatively different spatial behaviour, because of which they may be named *topological coherent modes*. Being the solutions of the nonlinear Schrödinger equation, these modes have nothing to do with collective excitations described by the linear Bogolubov - De Gennes equations.

Bose-Einstein condensation of trapped atoms can be understood [3] as the condensation to the ground coherent state, that is, to the state with the lowest single-particle energy. In an equilibrium system, atoms always condense to the ground state. But, if an additional alternating field is switched on, with a frequency being in resonance with the transition frequency between two coherent energy levels, then higher topological modes can be excited, thus creating nonground-state condensates. The feasibility of such modes was first advanced in Ref. [7]. The alternating resonant field can be produced by modulating the confining potential of the trap or by imposing additional external fields. For instance, a rotating field for exciting vortices, can be created by multiple laser beams [8–10]. The resonant excitation can lead to the formation of several vortices [11] and, possibly, skyrmions in a spinor condensate [12]. The topological coherent modes have also been studied in Refs. [13,14]. Excitation of a dipole mode in a two-component condensate was observed [15]. The possibility of creating dark soliton states by means of the resonant excitation was studied [11]. As is found, dark soliton states are unstable with respect to their decomposition into several vortices. The states of multiple basic vortices are also more stable than a state of a single vortex with a high winding number [3,16,17]. Investigating the temporal behaviour [18,19] and collective excitations [20] of the formed coherent modes, anomalous dynamical phenomena were found [18,19] reminding a kind of critical phenomena. The origin of the latter has not been completely understood.

The aim of the present paper is to develop the theory of the resonant formation of topological coherent modes and to provide an explanation of the dynamic critical phenomena. For this purpose, we give a rigorous stability analysis for the nonlinear evolution equations of guiding centers. We demonstrate that dramatic changes in the dynamic properties occur when crossing a saddle separatrix. The condition for the separatrix touching the boundary defines a critical line on the parametric manifold. This analysis is illustrated by the numerical solutions explicitly displaying a dramatic qualitative change in the dynamics of fractional populations when crossing the critical

line.

2 Resonant Excitation

Consider a system of Bose atoms, which have experienced Bose-Einstein condensation under conditions typical of experiments with trapped atomic gases [1–3]. The coherent state of such atoms is described [6] by a coherent field φ satisfying a nonlinear Schrödinger equation with the nonlinear Hamiltonian

$$\hat{H}(\varphi) = -\frac{\hbar^2}{2m_0} \nabla^2 + U(\mathbf{r}) + NA|\varphi|^2, \quad (1)$$

where m_0 is atomic mass, $U(\mathbf{r})$ is a confining potential, N is the number of trapped atoms, and $A \equiv 4\pi\hbar^2 a_s/m_0$ is the interaction strength, with a_s being an s -wave scattering length.

Topological coherent modes are defined [7] as stationary solutions $\varphi_n = \varphi_n(\mathbf{r})$ of the nonlinear Schrödinger equation

$$\hat{H}(\varphi_n)\varphi_n = E_n\varphi_n. \quad (2)$$

Here n is a multi-index labelling the quantum states of coherent modes, and E_n is a single-particle energy of the given coherent mode, which is normalized to unity as $(\varphi_n, \varphi_n) = 1$. The existence of the confining potential $U(\mathbf{r})$ assumes that the spectrum $\{E_n\}$ is discrete, hence the set $\{\varphi_n\}$ of eigenfunctions is countable.

It is worth mentioning that the solutions to a nonlinear Schrödinger equation do not necessarily form a basis and, in general, are not orthogonal. The property of being a basis has been proved for the eigenfunction sets of only some one-dimensional nonlinear problems [21,22]. However, it is important to stress that we do not need and shall not use these properties in what follows.

In order to induce intermode transitions, one has to impose an additional time-dependent potential $\hat{V} = \hat{V}(\mathbf{r}, t)$ and to consider a coherent field $\varphi = \varphi(\mathbf{r}, t)$ satisfying the temporal nonlinear Schrödinger equation

$$i\hbar \frac{\partial \varphi}{\partial t} = [\hat{H}(\varphi) + \hat{V}] \varphi. \quad (3)$$

Supposing that at the initial time all atoms were condensed to the ground-state coherent mode, one has the initial condition

$$\varphi(\mathbf{r}, 0) = \varphi_0(\mathbf{r}). \quad (4)$$

For transferring atoms from the ground state to a higher mode requires to impose an alternating potential

$$\hat{V} = V_1(\mathbf{r}) \cos \omega t + V_2(\mathbf{r}) \sin \omega t, \quad (5)$$

with a frequency ω being in resonance with the chosen transition frequency. If the wanted excited mode has the energy E_k , the transition frequency is

$$\omega_k \equiv \frac{1}{\hbar} (E_k - E_0). \quad (6)$$

Then the resonance condition tells that the detuning of ω from ω_k is to be small,

$$\left| \frac{\Delta\omega}{\omega} \right| \ll 1 \quad (\Delta\omega \equiv \omega - \omega_k) . \quad (7)$$

One expects that, under the resonance condition (7), only the considered modes, connected by the resonance frequency (6), will be mainly involved in the process of excitation. This can be proved explicitly by invoking the method of averaging [23]. For this purpose, let us look for the solution of equation (3) in the form

$$\varphi(\mathbf{r}, t) = \sum_n c_n(t) \varphi_n(\mathbf{r}) \exp\left(-\frac{i}{\hbar} E_n t\right) , \quad (8)$$

where $c_n(t)$ is a slowly varying amplitude, such that

$$\frac{\hbar}{E_n} \left| \frac{dc_n}{dt} \right| \ll 1 . \quad (9)$$

The presentation (8) is to be substituted into equation (3), which is multiplied by $\exp(\frac{i}{\hbar} E_n t)$ and whose right-hand side is averaged over time. Integrating over time, the amplitudes c_n , according to condition (9), are treated as quasi-invariants. The integration of exponents yields

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \exp\left\{\frac{i}{\hbar} (E_m - E_n)t\right\} dt &= \delta_{mn} , \\ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \exp\left\{\frac{i}{\hbar} (E_m + E_k - E_n - E_l)t\right\} dt &= \delta_{mn}\delta_{kl} + \delta_{ml}\delta_{nk} - \delta_{mk}\delta_{ln}\delta_{kl} . \end{aligned}$$

This procedure results in the equation

$$\frac{dc_n}{dt} = -i \sum_{m(\neq n)} \alpha_{nm} |c_m|^2 c_n - \frac{i}{2} \delta_{n0} \beta_{0k} c_k e^{i\Delta\omega t} - \frac{i}{2} \delta_{nk} \beta_{0k}^* c_0 e^{-i\Delta\omega t} , \quad (10)$$

in which the notation is used for the nonlinear transition amplitude

$$\alpha_{nm} \equiv A \frac{N}{\hbar} \int |\varphi_n(\mathbf{r})|^2 (2|\varphi_m(\mathbf{r})|^2 - |\varphi_n(\mathbf{r})|^2) d\mathbf{r} , \quad (11)$$

due to atomic interactions, and for the linear amplitude

$$\beta_{0k} \equiv \frac{1}{\hbar} \int \varphi_0^*(\mathbf{r}) [V_1(\mathbf{r}) - iV_2(\mathbf{r})] \varphi_k(\mathbf{r}) d\mathbf{r} , \quad (12)$$

caused by the resonant field (5). Note that the orthogonality of the modes $\varphi_n(\mathbf{r})$ has nowhere been required. The initial condition to equation (10), in line with condition (4), is

$$c_n(0) = \delta_{n0} . \quad (13)$$

In the case when $n \neq 0, k$, the solution to equation (10) can be written as

$$c_n(t) = c_n(0) \exp\left\{-i \sum_{m(\neq n)} \alpha_{nm} \int_0^t |c_m(t')|^2 dt'\right\} .$$

This, in compliance with the initial condition (13), shows that $c_n(t) = 0$ for all $n \neq 0, k$. So that only the behaviour of $c_0(t)$ and $c_k(t)$ is nontrivial, with the related initial conditions

$$c_0(0) = 1, \quad c_k(0) = 0. \quad (14)$$

Thus, equation (10) reduces to the system of equations

$$\begin{aligned} \frac{dc_0}{dt} &= -i\alpha_{0k}|c_k|^2 c_0 - \frac{i}{2} \beta_{0k} c_k e^{i\Delta\omega t}, \\ \frac{dc_k}{dt} &= -i\alpha_{k0}|c_0|^2 c_k - \frac{i}{2} \beta_{0k}^* c_0 e^{-i\Delta\omega t}, \end{aligned} \quad (15)$$

with the initial conditions (14). The solutions to equations (15) are called guiding centers.

3 Change of Variables

Equations (15) can be simplified by changing the variables. To this end, it is convenient to introduce the population difference

$$s \equiv |c_k|^2 - |c_0|^2. \quad (16)$$

The amplitudes c_0 and c_k can be presented as

$$\begin{aligned} c_0 &= \left(\frac{1-s}{2}\right)^{1/2} \exp\left\{i\left(\pi_0 + \frac{\Delta\omega}{2} t\right)\right\}, \\ c_k &= \left(\frac{1+s}{2}\right)^{1/2} \exp\left\{i\left(\pi_1 - \frac{\Delta\omega}{2} t\right)\right\}, \end{aligned} \quad (17)$$

with $\pi_0 = \pi_0(t)$ and $\pi_1 = \pi_1(t)$ being real phases.

Let us define the combination

$$\alpha \equiv \frac{1}{2} (\alpha_{0k} + \alpha_{k0}) \quad (18)$$

of the amplitudes (11), which is a real quantity, present the amplitude (12) as

$$\beta_{0k} \equiv \beta e^{i\gamma} \quad (\beta \equiv |\beta_{0k}|), \quad (19)$$

and also define the renormalized detuning

$$\delta \equiv \Delta\omega + \frac{1}{2} (\alpha_{0k} - \alpha_{k0}). \quad (20)$$

Lastly, we introduce the phase variable

$$x \equiv \pi_1 - \pi_0 + \gamma. \quad (21)$$

The new functional variables (16) and (21) change in the rectangle

$$-1 \leq s \leq 1, \quad 0 \leq x \leq 2\pi. \quad (22)$$

The related initial conditions are

$$s(0) = -1, \quad x(0) = x_0. \quad (23)$$

The value $x_0 = \pi_1(0) - \pi_0(0) + \gamma$ can be any in the interval $[0, 2\pi]$. This is because, even if the initial phases of the considered modes were the same, the quantity γ can be varied by choosing an appropriate alternating potential (5). Also, even when $\pi_1(0) = \pi_0(0)$, the time dependence of $\pi_1(t)$ and $\pi_0(t)$ is different, so that the evolution of $x = x(t)$ is not trivial.

With the new variables (16) and (21), equations (15) can be transformed to the Hamiltonian form characterized by the Hamiltonian

$$H(s, x) = \frac{\alpha}{2} s^2 - \beta \sqrt{1 - s^2} \cos x + \delta s. \quad (24)$$

The autonomous equations

$$\frac{ds}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dx}{dt} = \frac{\partial H}{\partial s}$$

are identical to those that follow from equations (15) after substituting there expressions (17), and which are

$$\frac{ds}{dt} = -\beta \sqrt{1 - s^2} \sin x, \quad \frac{dx}{dt} = \alpha s + \frac{\beta s}{\sqrt{1 - s^2}} \cos x + \delta. \quad (25)$$

Equations (25) are more convenient for analyzing than those (15).

The autonomous Hamiltonian form of the evolution equations (25) tells us two things. First, there is no dissipation for the coherent modes. The absence of intrinsic decoherence, connected with dissipation, here is quite clear. For such a decoherence to arise, the Bose system is to be in contact with an external bath [24,25] whose role can be played, e.g., by largely detuned external laser beams [26] or by disturbing measurement instruments [27]. Decoherence of a coherent wave packet may appear owing to the number-of-particle fluctuations [28], which, actually, presupposes the existence of an external bath. The latter is a general cause of dissipation for statistical systems [29,30] (other references on dissipative systems, can be found in the recent review [31]).

Another conclusion which results from the Hamiltonian representation of the autonomous evolution equations (25) is that in this dynamical system there can be no chaos. Hence the critical phenomena discovered [18,19] in the dynamics of fractional populations cannot be attributed to the appearance of chaotic motion. The origin of these critical phenomena will be elucidated in the following section.

4 General Analysis

To understand what happens with the solutions to the evolution equations (25) under varying parameters, we have to analyze the general phase structure of these equations. For simplifying formulas, it is useful to introduce the dimensionless parameters

$$b \equiv \frac{\beta}{\alpha}, \quad \varepsilon \equiv \frac{\delta}{\alpha}. \quad (26)$$

The first of the latter describes a relative intensity of the alternating field (5), while the second parameter characterizes a relative value of the detuning (20). These parameters can be positive as well as negative. The sole thing is that the relative detuning will be treated as a small parameter, $|\varepsilon| \ll 1$.

Due to the existence of the integral of motion (24), the trajectory, starting at the initial point (23) and defined by the equality $H(s, x) = H(s_0, x_0)$, writes

$$\frac{s^2}{2} - b\sqrt{1-s^2} \cos x + \varepsilon s = \frac{1}{2} - \varepsilon . \quad (27)$$

The right-hand sides of equations (25), with the notation (26), can be presented as

$$v_1 \equiv -b\sqrt{1-s^2} \sin x , \quad v_2 \equiv s + \frac{bs}{\sqrt{1-s^2}} \cos x + \varepsilon . \quad (28)$$

The stationary solutions to equations (25), given by $v_1 = v_2 = 0$, are defined by the equations

$$s^4 + 2\varepsilon s^3 - (1 - b^2 - \varepsilon^2)s^2 - 2\varepsilon s - \varepsilon^2 = 0 , \quad \sin x = 0 . \quad (29)$$

The following analysis, keeping in mind the smallness of the detuning, will be accomplished in the linear approximation with respect to ε . Also, one has to always remember that physically admissible fixed-point solutions are only those that are in the rectangle (22).

When $b^2 \geq 1$, there exist the fixed points

$$\begin{aligned} s_1^* &= \frac{\varepsilon}{b} , & x_1^* &= 0 , \\ s_2^* &= -\frac{\varepsilon}{b} , & x_2^* &= \pi , \\ s_3^* &= s_1^* , & x_3^* &= 2\pi . \end{aligned} \quad (30)$$

If $0 \leq b < 1$, the fixed points (30) continue to exist, but there arise two new points

$$\begin{aligned} s_4^* &= \sqrt{1-b^2} + \frac{b^2\varepsilon}{1-b^2} , & x_4^* &= \pi , \\ s_5^* &= -\sqrt{1-b^2} + \frac{b^2\varepsilon}{1-b^2} , & x_5^* &= \pi . \end{aligned} \quad (31)$$

And if $-1 < b \leq 0$, then the fixed points (30) again exist, but there appear the additional points

$$\begin{aligned} s_4^* &= \sqrt{1-b^2} + \frac{b^2\varepsilon}{1-b^2} , & x_4^* &= 0 , \\ s_5^* &= -\sqrt{1-b^2} + \frac{b^2\varepsilon}{1-b^2} , & x_5^* &= 0 , \\ s_6^* &= s_4^* , & x_6^* &= 2\pi , \\ s_7^* &= s_5^* , & x_7^* &= 2\pi . \end{aligned} \quad (32)$$

In this way, the value $b^2 = 1$ corresponds to a dynamical phase transition, when the structure of the phase portrait changes qualitatively.

To analyse the stability of motion, we consider the Jacobian expansion matrix $\hat{X} = [X_{ij}]$ with the elements

$$\begin{aligned} X_{11} &\equiv \frac{\partial v_1}{\partial s} = \frac{bs}{\sqrt{1-s^2}} \sin x, & X_{12} &\equiv \frac{\partial v_1}{\partial x} = -b\sqrt{1-s^2} \cos x, \\ X_{21} &\equiv \frac{\partial v_2}{\partial s} = 1 + \frac{b \cos x}{(1-s^2)^{3/2}}, & X_{22} &\equiv \frac{\partial v_2}{\partial x} = -X_{11}. \end{aligned}$$

The local expansion rate [32], defined as

$$\Lambda(t) \equiv \frac{1}{t} \operatorname{Re} \int_0^t \operatorname{Tr} \hat{X}(t') dt',$$

nullifies, $\Lambda(t) = 0$, as it should be for Hamiltonian systems whose phase volume is conserved. The eigenvalues of the expansion matrix \hat{X} are given by the equation

$$X^2 = \frac{b^2}{1-s^2} (s^2 \sin^2 x - \cos^2 x) - b\sqrt{1-s^2} \cos x. \quad (33)$$

Evaluating the eigenvalues at the fixed points, we shall employ expressions (30) to (32), limiting ourselves by the linear in ε approximation.

In the case of $b^2 > 1$, we have

$$X_1^\pm = \pm i\sqrt{b(b+1)} = X_3^\pm, \quad X_2^\pm = \pm i\sqrt{b(b-1)}, \quad (34)$$

so that all fixed points (30) are centers.

When $0 \leq b < 1$, the first and third fixed points remain the centers, while the second fixed point (s_2^*, x_2^*) becomes a saddle. The fixed points (31) are also the centers. The related exponents are

$$\begin{aligned} X_1^\pm &= \pm i\sqrt{b(1+b)} = X_3^\pm, & X_2^\pm &= \pm \sqrt{b(1-b)}, \\ X_4^\pm &= \pm i\sqrt{1-b^2} \left[1 + \frac{(2+b^2)\varepsilon}{(1-b^2)^{3/2}} \right]^{1/2}, & X_5^\pm &= \pm i\sqrt{1-b^2} \left[1 - \frac{(2+b^2)\varepsilon}{(1-b^2)^{3/2}} \right]^{1/2}. \end{aligned} \quad (35)$$

The saddle separatrices which are the trajectories that pass through the saddles and separate the phase regions with qualitatively different dynamic properties, are given by the equation $H(s, x) = H(s_2^*, x_2^*)$, which results in the separatrix equation

$$\frac{s^2}{2} - b\sqrt{1-s^2} \cos x + \varepsilon s = b, \quad (36)$$

defining two separatrices, since this is a square equation with respect to s . The separatrix extremal points can be found from the equation

$$\frac{ds}{dx} = - \frac{b(1-s^2) \sin x}{(s+\varepsilon)\sqrt{1-s^2} + bs \cos x} = 0.$$

As follows from the above equations, the lower separatrix parts touch the boundary at the points $s = -1$, $x = 0, 2\pi$, provided that

$$b + \varepsilon = \frac{1}{2} . \quad (37)$$

In this way, accepting as initial conditions $s_0 = -1$, $x_0 = 0$, one has the following picture. For $b < 0.5 - \varepsilon$, the motion is limited from above by the lower separatrix parts and from below, by the boundary $s = -1$. When the pumping parameter $b > 0.5 - \varepsilon$, the trajectory passes to the phase region limited from above by the upper separatrix parts and from below by the lower separatrix parts. If $b = 0.5 - \varepsilon$, the dynamical system is structurally unstable with respect to small variations of initial conditions. On the manifold of the system parameters, the line (37) plays the role of a *critical line*. In the vicinity of this line, solutions to the evolution equations display dramatic effects, when a tiny variation of a parameter qualitatively changes the properties of solutions. This makes it possible to call such dynamical effects the *critical dynamic phenomena* [33].

For the case when $-1 < b \leq 0$, the expansion exponents, corresponding to the fixed points (30) and (32), are

$$\begin{aligned} X_1^\pm &= \pm \sqrt{|b|(1+b)} = X_3^\pm , & X_2^\pm &= \pm i \sqrt{|b|(1-b)} , \\ X_4^\pm &= \pm i \sqrt{1-b^2} \left[1 + \frac{(2+b^2)\varepsilon}{(1-b^2)^{3/2}} \right]^{1/2} = X_6^\pm , \\ X_5^\pm &= \pm i \sqrt{1-b^2} \left[1 - \frac{(2+b^2)\varepsilon}{(1-b^2)^{3/2}} \right]^{1/2} = X_7^\pm . \end{aligned} \quad (38)$$

Hence, the first and third fixed points become the saddles, while all other points are the centers. The separatrices connecting the saddles are defined by the equations $H(s, x) = H(s_1^*, x_1^*) = H(s_3^*, x_3^*)$, which yield

$$\frac{s^2}{2} - b\sqrt{1-s^2} \cos x + \varepsilon s = -b . \quad (39)$$

The lower separatrix part touches the boundary at the point $s = -1$, $x = \pi$ under the condition

$$|b| + \varepsilon = \frac{1}{2} . \quad (40)$$

The phase picture, as compared to the previous case $0 \leq b < 1$, looks similar, but being shifted by π along the axis x . Now, if the initial point would be $s_0 = -1$, $x_0 = 0$, the motion would be always limited from above by the lower separatrix parts and from below, by the boundary $s = -1$. No dramatic changes would happen when varying b . However, if the initial point is taken as $s_0 = -1$, $x_0 = \pi$, one again encounters the same critical dynamic phenomena on the critical line (40).

This analysis explains that the occurrence of critical dynamic phenomena is caused by the existence of a critical line on the manifold of system parameters and happens only under a special choice of initial conditions, when the latter are touched by a

separatrix. The initial conditions for the variable (21) can be varied in a wide diapason by choosing the appropriate alternating field (5), which would yield the related linear amplitude (12), with the required value of

$$\gamma = \arg \beta_{0k} = \tan^{-1} \frac{\text{Im } \beta_{0k}}{\text{Re } \beta_{0k}},$$

defined in equation (19).

5 Numerical Solution

In order to explicitly illustrate the critical dynamic phenomena, occurring when crossing the critical line on the parametric manifold, we have accomplished numerical calculations of the fractional populations

$$n_0(t) \equiv |c_0(t)|^2 = \frac{1 - s(t)}{2}, \quad n_k(t) \equiv |c_k(t)|^2 = \frac{1 + s(t)}{2}.$$

This can be done by numerically solving either equations (15) or (25), which are equivalent. The results are presented in figures 1 to 3, where time is measured in units of α^{-1} , the pumping parameter $b = 0.5$ is fixed, while the detuning ε is varied so that to cross the critical line (37). The initial conditions for the evolution equations are taken as $s_0 = -1$, $x_0 = 0$, that is $n_0(0) = 1$, $n_k(0) = 0$.

In figure 1, the detuning is negative, so that one is slightly below the critical line (37). The fractional populations oscillate displaying a kind of Rabi oscillations, if one looks for analogies with optics [34]. However, since the problem considered here is nonlinear, there is no a well defined constant Rabi frequency, whose analog would be now a function of time [18,19]. In some sense, these oscillations could be named *non-linear Rabi oscillations*. Approaching the critical line (37) by increasing the detuning ε , the oscillation amplitude increases. The motion of the population n_k of the excited topological mode is limited from above by the lower parts of separatrices and from below, by the boundary $n_k = 0$.

Figure 2 demonstrates the motion in a tiny vicinity of the critical line (37). Changing the detuning from $\varepsilon = 10^{-4}$ to $\varepsilon = 0$ drastically transforms the shape of oscillations. The oscillation period more than doubles and wide flat platos in the time dependence of the population appear. Crossing the critical line, with a microscopic variation of the detuning from $\varepsilon = 0$ to $\varepsilon = 10^{-9}$, again yields a drastic transformation of the population shapes. The period is again almost doubled; the upward casps of $n_k(t)$ and down-ward casps of $n_0(t)$ arise. The appearance of these casps means that the motion has passed to another phase region, as has been discussed in the analysis of the previous section.

After crossing the critical line (37), the dynamics of the fractional populations remains qualitatively unchanged. Increasing the detuning slightly changes the oscillation period and smoothes the shape of the oscillation curves, as is shown in figure 3.

In conclusion, we have presented a theory for the resonant excitation of *topological coherent modes* of trapped Bose atoms. These modes are the stationary solutions to

the nonlinear Schrödinger equation, which is also sometimes called the Gross-Pitaevskii equation. This equation provides a correct description of trapped atoms at low temperatures [1–3,35]. The principally important part of the paper is the demonstration of the occurrence of *critical dynamic phenomena* in the process of exciting coherent modes and a thorough elucidation of the origin of these phenomena.

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Figure Captions

Figure 1. The fractional populations of the ground coherent mode $n_0(t)$ (dashed line) and of the excited topological coherent mode $n_k(t)$ (solid line) as functions of time, measured in units of α^{-1} , for the fixed pumping parameter $b = 0.5$ and the negative detuning parameter below the critical line: (a) $\varepsilon = -10^{-1}$; (b) $\varepsilon = -10^{-2}$; (c) $\varepsilon = -10^{-4}$.

Figure 2. Dramatic changes in the dynamics of the fractional populations of the ground coherent mode (dashed line) and excited mode (solid line) when crossing the critical line on the parametric manifold for fixed $b = 0.5$ and varying detuning: (a) $\varepsilon = 0$; (b) $\varepsilon = 10^{-9}$.

Figure 3. Qualitatively different temporal behaviour of the fractional populations of the ground mode (dashed line) and excited mode (solid line) after crossing the parametric critical line, with fixed $b = 0.5$ and varying detuning: (a) $\varepsilon = 10^{-4}$; (b) $\varepsilon = 10^{-2}$; (c) $\varepsilon = 10^{-1}$.